

# ON SELF-ASSOCIATED SETS OF POINTS IN SMALL PROJECTIVE SPACES

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**ABSTRACT.** We study moduli of “self-associated” sets of points in  $\mathbf{P}^n$  for small  $n$ . In particular, we show that for  $n = 5$  a general such set arises as a hyperplane section of the Lagrangean Grassmanian  $LG(5, 10) \subset \mathbf{P}^{15}$  (this was conjectured by Eisenbud-Popescu in *Geometry of the Gale transform*, J. Algebra 230); for  $n = 6$ , a general such set arises as a hyperplane section of the Grassmanian  $G(2, 6) \subset \mathbf{P}^{14}$ . We also make a conjecture for the next case  $n = 7$ . Our results are analogues of Mukai’s characterization of general canonically embedded curves in  $\mathbf{P}^6$  and  $\mathbf{P}^7$ , resp.

## 1. INTRODUCTION

Let  $\Gamma \subset \mathbf{P}^n = \mathbf{P}V$  be a nondegenerate set of  $2n + 2$  distinct points with the following property: one can write  $\Gamma = \Gamma' \cup \Gamma''$  such that both  $\Gamma'$  and  $\Gamma''$  correspond to orthogonal bases for some nondegenerate bilinear form  $Q$  on  $V$  (i.e. both  $\Gamma'$  and  $\Gamma''$  are *apolar* simplices with respect to  $Q$ ). Then, we say that  $\Gamma$  is a *self-associated (s.a.) set of points* in  $\mathbf{P}^n$ . This notion of self-association was first introduced by Castelnuovo [4] and further studied by Coble ([6], [7]) and other classical geometers ([1], [2]). The theory of self-association was given a modern treatment by Dolgachev-Ortland ([8]) and, more recently, by Eisenbud-Popescu ([9]) in connection with the minimal resolution conjecture. Self-associated sets of points were also studied by Schreyer-Tonoli ([19]) in connection with the Green’s conjecture on syzygies of canonical curves.

An equivalent definition of self-association is the following one (see [9], thm. 7.1 and 8.1):

**Definition 1.1.** Let  $\Gamma \subset \mathbf{P}^n$  be a nondegenerate set of  $2n + 2$  distinct points in  $\mathbf{P}^n$ . Then,  $\Gamma$  is s.a. set of points if and only if any subset of  $2n + 1$  points of  $\Gamma$  imposes the same number of conditions on quadrics as  $\Gamma$ .

There is a moduli space, which we denote  $\mathcal{A}_n$ , parametrizing (unordered) self-associated sets of points in  $\mathbf{P}^n$  modulo projective equivalence. It is shown in [8], that  $\mathcal{A}_n$  is an irreducible, unirational variety of dimension  $\binom{n+1}{2}$ .

A more sophisticated characterization of self-association is provided by the following theorem:

**Theorem 1.2.** ([9]) *Let  $\Gamma$  be a nondegenerate set of  $2n + 2$  points in  $\mathbf{P}^n$ . Then,  $\Gamma$  is Arithmetically Gorenstein (AG) if and only if  $\Gamma$  is self-associated and  $\Gamma$  fails exactly by 1 to impose independent conditions on quadrics.*

It is easy to see, that if  $\Gamma$  is a sufficiently general s.a. set of points (e.g.  $\Gamma$  is in linearly general position), then the theorem above applies to  $\Gamma$ .

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**Example 1.3.** If  $\Gamma$  is a set of  $2n + 2$  points that lies on a rational normal curve  $D \subset \mathbf{P}^n$ , then  $\Gamma$  is a s.a. set of points.

**Example 1.4.** Let  $C \subset \mathbf{P}^{n+1}$  be a smooth canonically embedded curve. It is well-known, that  $C$  is AG. Therefore, any transversal hyperplane section  $\Gamma = C \cap \mathbf{P}^n$  is AG, and so, it forms a s.a. set of points in  $\mathbf{P}^n$  (see [19]).

A geometric characterization of s.a. sets of points is classically known in small dimensions (see [9], sec. 9). Namely, in  $\mathbf{P}^2$ , a s.a. set of points  $\Gamma$  is just a complete intersection of a conic and a cubic. In  $\mathbf{P}^3$ , a sufficiently general s.a. set of points is a complete intersection of three quadrics. In  $\mathbf{P}^4$ , a sufficiently general s.a. set of points is a complete intersection of an elliptic normal curve  $E \subset \mathbf{P}^4$  and a quadric. In particular, in all three cases,  $\Gamma$  arises as a hyperplane section of a canonical curve.

The goal of the present work is to give a similar characterization of s.a. sets of points in dimensions  $\mathbf{P}^5$  and  $\mathbf{P}^6$ . More precisely, we will show, that a general s.a. set of points in these dimensions arises as a hyperplane section of a smooth canonical curve. In fact, our result for the case  $\mathbf{P}^5$  answers a conjecture of Eisenbud-Popescu ([9]). On the other hand, the pattern ceases to generalize in dimension  $\mathbf{P}^7$ .

The starting point is the well-known theorem of Mukai ([17]):

**Theorem 1.5.** (a) *A general canonical curve  $C \subset \mathbf{P}^6$  of genus  $g = 7$  is a linear section of the Lagrangean Grassmanian  $LG_+(5, 10) \subset \mathbf{P}^{15}$ ;*  
 (b) *A general canonical curve  $C \subset \mathbf{P}^7$  of genus  $g = 8$  is a linear section of the Grassmanian  $G(2, 6) \subset \mathbf{P}^{14}$ ;*  
 (c) *A general canonical curve  $C \subset \mathbf{P}^8$  of genus  $g = 9$  is a linear section of the symplectic Grassmanian  $G_\omega(3, 6) \subset \mathbf{P}^{13}$ .*

We will show that parts (a) and (b) of the theorem above extend to s.a. sets of points in  $\mathbf{P}^5$  and  $\mathbf{P}^6$ , resp. On the other hand, the analogue of part (c) fails for s.a. sets of points in  $\mathbf{P}^7$  because of a simple moduli count (see conjecture 2.12).

Ranestad-Schreyer ([18]) proved that “a general empty AG scheme of degree 12 in  $\mathbf{P}^4$ ” (that is, a graded Artinian Gorenstein ring with Hilbert function  $(1, 5, 5, 1)$ ) arises as a linear section of  $LG_+(5, 10)$ . Thus, our result for s.a. sets of points in  $\mathbf{P}^5$  should be viewed as an intermediate step between their result and that of Mukai.

Also, we reinterpret our result for s.a. sets of points in  $\mathbf{P}^5$  as an instance of “inversion” of the Cayley-Bacharach theorem for vector bundles ([12], [14]).

*Notation and conventions.* We work over a base field  $k$  of characteristic 0. For any closed reduced subscheme  $X \subset \mathbf{P}^n$  with ideal sheaf  $\mathcal{I}_X$ , and any positive integer  $m$ , denote by  $X^{(m)} \subset \mathbf{P}^n$  the subscheme defined by the ideal sheaf  $\mathcal{I}_X^{\otimes m}$ . In other words,  $X^{(m)}$  the smallest scheme supported on  $X$  that contains the  $(m - 1)$ -st infinitesimal neighborhood of  $X$ . For any  $r < n$ , denote by  $G(r, n) = G(\mathbf{P}^{r-1}, \mathbf{P}^{n-1})$  the Grassmanian of  $(r - 1)$ -planes in  $\mathbf{P}^{n-1}$ .

## 2. CHARACTERIZATION OF S.A. SETS OF POINTS IN SMALL DIMENSIONS

2.1. The case  $\mathbf{P}^5$ .

Recall some standard facts about Lagrangean Grassmanians ([16], [11]). Let  $(V, Q)$  be a  $2n$ -dimensional vector space  $V$  equipped with a nondegenerate symmetric bilinear form  $Q$ . A linear subspace  $W \subset V$  is *isotropic* iff  $Q(w, w) = 0$  for any  $w \in W$ . An isotropic subspace of maximal dimension  $n$  is called a *Lagrangean* subspace. Since we assume the dimension of  $V$  to be even, the variety parametrizing all such subspaces consists of two disjoint components  $LG_{\pm}(n, V)$ .

Write  $\wedge^{\bullet}V = \wedge^{ev}V \oplus \wedge^{odd}V$ . Then, the Lagrangean Grassmanian  $LG_{+}(n, V)$  (resp.  $LG_{-}(n, V)$ ) admits the Plücker embedding  $LG_{+}(n, V) \hookrightarrow \mathbf{P}(\wedge^{ev}V)$  (resp.  $LG_{-}(n, V) \hookrightarrow \mathbf{P}(\wedge^{odd}V)$ ). The special orthogonal group  $SO(V, Q)$  acts on  $\wedge^{ev}V$  (resp.  $\wedge^{odd}V$ ) via the *(half-)spin representation* and  $LG_{+}(n, V)$  (resp.  $LG_{-}(n, V)$ ) is a homogeneous space for this representation.

Next, we specialize to the case  $\dim V = 10$ . Denote  $X = LG_{+}(5, 10)$ . One can show ([16]), that  $X \subset \mathbf{P}^{15}$  is a 10-dimensional variety of degree 12. Also, the canonical bundle  $K_X \cong \mathcal{O}_X(-8)$ .

It is observed in ([16]), that the homogeneous ideal of  $X \subset \mathbf{P}^{15}$  is generated by 10 quadrics, that in turn satisfy a unique quadratic relation. More precisely, consider the natural map

$$\rho : \text{Sym}^2 H^0(\mathbf{P}^{15}, \mathcal{I}_X(2)) \longrightarrow H^0(\mathbf{P}^{15}, \mathcal{I}_{X(2)}(4))$$

induced by multiplication. Then, the kernel of  $\rho$  is 1-dimensional, generated by a nondegenerate symmetric bilinear form, which we denote  $R^*$ . Taking the dual, the space of quadrics  $H^0(\mathbf{P}^{15}, \mathcal{I}_X(2))$  is naturally endowed with a nondegenerate symmetric bilinear form  $R$ .

Notice, that there is a natural action of  $SO(10)$  on  $H^0(\mathbf{P}^{15}, \mathcal{I}_X(2))$ , induced by the spin representation.

**Proposition 2.1.** (*Mukai, [16]*) *There is a natural isomorphism*

$$\alpha : V \xrightarrow{\sim} H^0(\mathbf{P}^{15}, \mathcal{I}_X(2))$$

*with the following properties:*

- (a)  $\alpha$  is  $SO(10)$ -equivariant and  $\alpha^*(R) = Q$ .
- (b) Let  $x \in X$  be a point, corresponding to a Lagrangean  $W_x \subset V$ . Then,  $\alpha(W_x) \subset H^0(\mathbf{P}^{15}, \mathcal{I}_X(2))$  is precisely the linear subsystem of quadrics containing  $X$  that are singular at  $x$ .

Let  $X = LG_{+}(5, 10)$  as above. Since  $K_X \cong \mathcal{O}_X(-8)$  and  $X$  is projectively normal, it follows, that a general linear section  $\Gamma = X \cap \mathbf{P}^5$  is a s.a. set of points in  $\mathbf{P}^5$ . It was conjectured by Eisenbud-Popescu in [9], that any sufficiently general s.a. set of points in  $\mathbf{P}^5$  arises in this way. Indeed, we have:

**Theorem 2.2.** *Let  $\Gamma$  be a general s.a. set of points in  $\mathbf{P}^5$ . Then,  $\Gamma$  is a linear section of the Lagrangean Grassmanian  $LG_{+}(5, 10) \subset \mathbf{P}^{15}$ . Moreover, the linear section is unique upto the natural  $SO(10)$ -action induced by the spin representation.*

From Mukai's result, we conclude:

**Corollary 2.3.** *A general s.a. set of points in  $\mathbf{P}^5$  is a linear section of a canonical curve in  $\mathbf{P}^6$ .*

Our approach is essentially a translation of Mukai's proof from curves to points. A certain geometric argument of Mukai is replaced by a symbolic computation on the computer algebra system *Macaulay 2* ([3]).

*Proof of Theorem 2.2.* Let  $\Xi \subset G(\mathbf{P}^5, \mathbf{P}^{15})$  be the open subset of linear subspaces that meet  $X \subset \mathbf{P}^{15}$  transversally at 12 distinct points. Notice, that  $SO(10)$  acts on  $\Xi$  and there is a well-defined morphism

$$\sigma : \Xi/SO(10) \longrightarrow \mathcal{A}_5,$$

that assigns to every  $[\Lambda] \in \Xi$  the s.a. set of points  $X \cap \Lambda$ . By counting moduli, we get:

$$\dim \Xi/SO(10) = 6(15 - 5) - \binom{10}{2} = 15$$

and

$$\dim \mathcal{A}_5 = \binom{5+1}{2} = 15.$$

Therefore, to show that  $\sigma$  is birational, it suffices to show that  $\sigma$  is injective.

We will need the following result.

**Lemma 2.4.** *The Lagrangean Grassmanian  $X \subset \mathbf{P}^{15}$  is ACM and so is the scheme  $X^{(2)} \subset \mathbf{P}^{15}$ .*

*Proof.* Homogeneous varieties are known to be ACM in general. We have verified that  $X^{(2)}$  is ACM by using *Macaulay 2*. See the Appendix for details.  $\square$

**Corollary 2.5.** *Let  $\Gamma = X \cap \mathbf{P}^5$  be any transversal linear section. Consider the commutative diagram*

$$\begin{array}{ccc} \text{Sym}^2 H^0(\mathbf{P}^{15}, \mathcal{I}_X(2)) & \xrightarrow{\rho} & H^0(\mathbf{P}^{15}, \mathcal{I}_{X^{(2)}}(4)) \\ \downarrow & & \downarrow \\ \text{Sym}^2 H^0(\mathbf{P}^5, \mathcal{I}_\Gamma(2)) & \xrightarrow{\bar{\rho}} & H^0(\mathbf{P}^5, \mathcal{I}_{\Gamma^{(2)}}(4)) \end{array}$$

where the horizontal maps are induced by multiplication and the vertical maps are induced by restriction. Then, the two vertical maps are isomorphisms.  $\square$

We are ready to complete the proof of the theorem. Let  $\Gamma = X \cap \Lambda \subset \mathbf{P}^{15}$  be a transversal linear section with  $\Lambda \cong \mathbf{P}^5$ . By cor. 2.5, the map

$$\bar{\rho} : \text{Sym}^2 H^0(\mathbf{P}^5, \mathcal{I}_\Gamma(2)) \rightarrow H^0(\mathbf{P}^5, \mathcal{I}_{\Gamma^{(2)}}(4))$$

has a 1-dimensional kernel, generated by a nondegenerate symmetric bilinear form  $\bar{R}^*$ . Let  $\bar{R}$  be the dual of  $\bar{R}^*$ .

[Here is a heuristic way to see why  $\dim \ker \bar{\rho} = 1$ . Clearly,  $\dim \text{Sym}^2 H^0(\mathbf{P}^5, \mathcal{I}_\Gamma(2)) = 55$ . Also, we would have  $h^0(\mathbf{P}^5, \mathcal{I}_{\Gamma^{(2)}}(4)) = \binom{4+5}{5} - 12 \cdot 6 = 54$ , assuming that double points at  $\Gamma$  impose independent conditions on quartics in  $\mathbf{P}^5$ .]

Let

$$\beta : H^0(\mathbf{P}^{15}, \mathcal{I}_X(2)) \xrightarrow{\sim} H^0(\mathbf{P}^5, \mathcal{I}_\Gamma(2))$$

be the map, induced by restriction to  $\Lambda$ . By cor. 2.5, we have  $\beta^*(\bar{R}) = R$ . Composing with the map  $\alpha$  from prop. 2.1, we get an isomorphism

$$\beta \circ \alpha : V \xrightarrow{\sim} H^0(\mathbf{P}^5, \mathcal{I}_\Gamma(2))$$

such that  $(\beta \circ \alpha)^*(\bar{R}) = Q$ .

From now on, we identify  $V$  with  $H^0(\mathbf{P}^5, \mathcal{I}_\Gamma(2))$  via  $\beta \circ \alpha$ . For any  $p_i \in \Gamma$ , denote by  $W_i \subset V$  the linear subspace parametrizing quadrics containing  $\Gamma$  that are singular at  $p_i$ . Since  $\Gamma$  is cut-out scheme-theoretically by quadrics, a double point at  $p_i \in \Gamma$  imposes 5 conditions on quadrics in  $\Lambda$  passing through  $p_i$ . Therefore,  $\dim W_i = 5$ . It follows, that any such quadric in  $\Lambda$  is a restriction of a quadric in  $\mathbf{P}^{15}$  containing  $X$  and singular at  $p_i$ . By prop. 2.1,  $W_i \subset V$  is a Lagrangean and moreover, it corresponds to the point  $p_i \in \Gamma \subset X$ , i.e.  $p_i = [W_i]$ .

Now, let  $\Gamma' = X \cap \Lambda'$  be another transversal linear section and assume, that  $\Gamma \subset \Lambda$  and  $\Gamma' \subset \Lambda'$  are projectively equivalent. We claim, that  $\Lambda$  and  $\Lambda'$  are in the same  $SO(10)$ -orbit.

For, let

$$\beta' : H^0(\mathbf{P}^{15}, \mathcal{I}_X(2)) \xrightarrow{\sim} H^0(\mathbf{P}^5, \mathcal{I}_{\Gamma'}(2))$$

be the map, induced by restriction to  $\Lambda'$ . Hence,  $\beta^*(\bar{R}) = \beta'^*(\bar{R}) = R$ . It follows, that  $\beta' = \beta \circ \tau$  for some  $\tau \in SO(10)$ . For any  $p'_i \in \Gamma' \subset X$ , define the Lagrangean  $W'_i \subset V$  as before. It follows, that  $[W'_i] = \tau[W_i]$ . Therefore,  $[\Lambda'] = \tau[\Lambda]$ .

This shows, that the map  $\sigma : \Xi/SO(10) \rightarrow \mathcal{A}_5$  is injective, which completes the proof.  $\square$

*Alternative interpretation.*

We interpret Theorem 2.2 as an instance of “inversion” of the Cayley-Bacharach theorem for vector bundles (see [12], [14]).

Consider a surjective map vector bundles

$$f : \mathcal{O}_{\mathbf{P}^5}(2)^7 \rightarrow \mathcal{O}_{\mathbf{P}^5}(3)^2$$

and let  $B_f = \ker f$ . If  $f$  is sufficiently general, then  $B_f$  is a vector bundle of rank 5 on  $\mathbf{P}^5$ . By Whitney’s product formula, we have:

$$\begin{aligned} c_1(B_f) &= 8[H]; \\ c_5(B_f) &= 12[H]^5. \end{aligned}$$

Here,  $[H]$  denotes the hyperplane class.

Let  $\Gamma = Z(s)$  be the zero-locus of a regular section  $s \in H^0(B_f)$  and assume that  $\Gamma$  is reduced. By the Cayley-Bacharach theorem for vector bundles,  $\Gamma$  fails exactly by one to impose independent conditions on the linear system  $|K_{\mathbf{P}^5} \otimes \det(B_f)| = |\mathcal{O}_{\mathbf{P}^5}(2)|$ . In other words,  $\Gamma$  is a self-associated set of points in  $\mathbf{P}^5$ .

Remarkably, Theorem 2.2 implies (and, in fact, is equivalent to) the converse statement:

*If  $\Gamma$  is a general self-associated set of points in  $\mathbf{P}^5$ , then  $\Gamma$  is the zero locus of a regular section of a vector bundle  $B_f$ , for a suitable choice of  $f$ .*

We just sketch the proof. Consider a sufficiently general map of vector bundles

$$\tilde{f} : \mathcal{O}_{\mathbf{P}^{15}}(2)^7 \rightarrow \mathcal{O}_{\mathbf{P}^{15}}(3)^2$$

and let  $B_{\tilde{f}} = \ker \tilde{f}$ . Then,  $B_{\tilde{f}}$  is a reflexive sheaf of rank 5 (in fact, a Buchsbaum-Rim sheaf, in the language of [15]). Let  $s \in H^0(B_{\tilde{f}})$  be a general section. One can show, that the top component of the zero locus  $Z(s)$  is projectively equivalent to the Lagrangean Grassmanian  $LG_+(5, 10)$  ( $Z(s)$  also has a component of larger codimension, namely the locus of degeneracy of  $\tilde{f}$ ). We have checked this on

*Macaulay 2*. It remains to consider the restriction of  $B_{\tilde{f}}$  to a general  $\mathbf{P}^5 \subset \mathbf{P}^{15}$  and apply Theorem 2.2.

## 2.2. The case $\mathbf{P}^6$ .

Recall some standard facts about Grassmanians of lines ([13]). Let  $V$  be an  $n$ -dimensional vector space over the base field  $k$ . Consider the Grassmanian  $X = G(2, V) \subset \mathbf{P}(\wedge^2 V)$  parametrizing lines in  $\mathbf{P}V$ , together with its Plücker embedding. The group  $SL(V)$  naturally acts on  $\wedge^2 V$  and  $X$  is a homogeneous space for this action.

For any  $w \in \wedge^2 V$  and  $k \leq n/2$ , denote

$$\text{pf}(2k, w) = \underbrace{w \wedge \cdots \wedge w}_{k \text{ times}} \in \wedge^{2k} V.$$

If we fix a basis for  $V$ , then  $\text{pf}(2k, w)$  is just the  $2k \times 2k$ -pfaffians of the skew-symmetric matrix given by  $w$ .

It is well-known, that  $X \subset \mathbf{P}(\wedge^2 V)$  can be defined as the locus of vanishing of  $\text{pf}(4, w)$ , for  $w \in \wedge^2 V$ . More generally, for any  $m$ , the  $m$ -secant variety  $\sigma_m(X)$  can be defined as the locus of vanishing of  $\text{pf}(2m+4, w)$ .

Next, we specialize to the case  $\dim V = 6$ . Then,  $X = G(2, 6) \subset \mathbf{P}^{14}$  is an 8-dimensional variety of degree 14. Also, the canonical bundle  $K_X \cong \mathcal{O}_X(-6)$ . It follows, that a general section  $\Gamma = X \cap \mathbf{P}^6$  is a s.a. set of points in  $\mathbf{P}^6$ . Our main result in this section is the converse statement:

**Theorem 2.6.** *A general s.a. set of points in  $\mathbf{P}^6$  is a linear section of the Grassmanian  $G(2, 6) \subset \mathbf{P}^{14}$ .*

By Mukai's theorem, we have:

**Corollary 2.7.** *A general s.a. set of points in  $\mathbf{P}^6$  is a hyperplane section of a canonical curve in  $\mathbf{P}^7$ .*

*Proof of theorem 2.6.* Just as in the proof of theorem 2.2, we start with moduli count. Let  $\Xi \subset G(\mathbf{P}^6, \mathbf{P}^{14})$  be the open subset of linear subspaces that meet  $X \subset \mathbf{P}^{14}$  transversally at 14 distinct points. Now,  $SL(6)$  acts on  $\Xi$  and we have a well-defined map

$$\sigma : \Xi / SL(6) \rightarrow \mathcal{A}_6,$$

that assigns to every  $[\Lambda] \in \Xi$  the s.a. set of points  $X \cap \Lambda$ . We have:

$$\dim \Xi / SL(6) = 7(14 - 6) - 35 = 21$$

and

$$\dim \mathcal{A}_6 = \binom{6+1}{2} = 21.$$

To complete the proof of the theorem, we will show, that  $\sigma$  is generically finite.

We will need the following result.

**Lemma 2.8.** *The Grassmanian  $X = G(2, 6) \subset \mathbf{P}^{14}$  is ACM and so is the scheme  $X^{(2)}$ .*

*Proof.* Since  $X$  is homogeneous, it is ACM. We have verified the statement for  $X^{(2)}$  via *Macaulay 2* (See the Appendix for details).  $\square$

Consider the secant variety  $\sigma_1(X)$ , given by  $pf(6, w) = 0$ . Then,  $\sigma_1(X)$  is the unique cubic hypersurface in  $\mathbf{P}^{14}$  with singular locus  $X$ . Since  $X^{(2)}$  is ACM, we get:

**Corollary 2.9.** *Let  $\Gamma = X \cap \mathbf{P}^6$  is any transversal linear section. Then, there is a unique cubic hypersurface in  $\mathbf{P}^6$  that is singular at every point of  $\Gamma$ .*  $\square$

Let  $\Gamma = X \cap \Lambda$  for a sufficiently general  $[\Lambda] \in \Xi$ . Then,  $\Gamma$  is defined as the vanishing locus of the  $4 \times 4$  pfaffians in a  $6 \times 6$  skew-symmetric matrix  $A \in H^0(\wedge^2 V \otimes \mathcal{O}_{\mathbf{P}^6}(1))$ . In coordinate-free form,  $A$  is simply the restriction of the identity element  $1 \in H^0(\wedge^2 V \otimes \mathcal{O}_{\mathbf{P}(\wedge^2 V)}(1))$ . Conversely, the matrix  $A$  uniquely determines  $[\Lambda] \in \Xi$ .

Now, let  $\Gamma' = X \cap \Lambda'$  for some other  $[\Lambda'] \in \Xi$  and assume, that  $\Gamma' \subset \Lambda'$  is projectively equivalent to  $\Gamma \subset \Lambda$ . Let  $B \in H^0(\wedge^2 V \otimes \mathcal{O}_{\mathbf{P}^6}(1))$  be the corresponding skew-symmetric matrix. In what follows, we identify the ambient spaces of  $\Gamma$  and  $\Gamma'$ , and simply write  $\mathbf{P}^6$ . After normalizing  $B$  by a scalar if necessary, cor. 2.9 implies, that  $pf(6, A) = pf(6, B)$ . We have:

**Lemma 2.10.** *Let  $\Gamma$  and  $A \in H^0(\wedge^2 V \otimes \mathcal{O}_{\mathbf{P}^6}(1))$  be as above. Then, there is a Zariski open subset  $U \subseteq H^0(\wedge^2 V \otimes \mathcal{O}_{\mathbf{P}^6}(1))$ , such that the following is true: for any  $B \in U$ ,  $pf(6, A) = pf(6, B)$  if and only if  $A = B^\tau$  for some  $\tau \in SL(V)$ .*

(Here we denote  $A^\tau := \tau^t A \tau$ , which is independent of the choice of a basis for  $V$ .)

*Proof.* Since  $pf(6, A) = pf(6, A^\tau)$ , the “if” part is true for any open  $U$ .

Let  $Z \subset H^0(\wedge^2 V \otimes \mathcal{O}_{\mathbf{P}^6}(1))$  be the closed subset of elements  $B$  such that  $pf(6, A) = pf(6, B)$ , and let  $T_A Z$  be the Zariski tangent space to  $Z$  at  $A$ . Consider the map  $\rho : SL(V) \rightarrow Z$ , given by  $\tau \mapsto A^\tau$ , and its differential  $d\rho : \mathfrak{sl}(V) \rightarrow T_A Z$ . It is easy to see, that if  $A$  is sufficiently general, then  $d\rho$  is injective.

Next, we will show, that  $d\rho$  is in fact bijective. For any  $B' \in T_A Z$ , let  $B = A + \epsilon B'$ , where  $\epsilon^2 = 0$ . Since  $pf(B) = pf(A) + 3\epsilon B' \wedge A \wedge A$  and  $\text{char } k \neq 3$ , we have:

$$B' \wedge A \wedge A = 0.$$

This is a system of linear equations for  $B'$  which can be solved explicitly by looking at the resolution of  $\Gamma$ . Since  $X$  is ACM,  $\Gamma$  has the same resolution as  $X$  (see the Appendix):

$$\dots \longrightarrow \mathcal{O}_{\mathbf{P}^6}(-3)^{35} \longrightarrow \mathcal{O}_{\mathbf{P}^6}(-2)^{15} \xrightarrow{A \wedge A} \mathcal{O}_{\mathbf{P}^6} \longrightarrow \mathcal{O}_\Gamma \longrightarrow 0$$

Therefore,  $\dim T_A Z = \dim \mathfrak{sl}(V) = 35$ , and so  $d\rho : \mathfrak{sl}(V) \rightarrow T_A Z$  is bijective.

The lemma follows.  $\square$

From what we said, it follows that there is an open subset  $\tilde{U} \subset \Xi$  containing  $[\Lambda]$  such that the following is true: if  $[\Lambda'] \in \tilde{U}$  is such that  $\Gamma' = X \cap \Lambda'$  is projectively equivalent to  $\Gamma$ , then  $[\Lambda'] = \tau[\Lambda]$ , for some  $\tau \in SL(V)$ .

This implies, that the map  $\sigma : \Xi/SL(V) \rightarrow \mathcal{A}_6$  is generically finite, which completes the proof of the theorem.  $\square$

**Remark 2.11.** We expect, that the map  $\sigma : \Xi/SL(V) \rightarrow \mathcal{A}_6$  is actually birational. This would be true, if we knew that lemma 2.10 holds globally, i.e. we may take  $U = \Gamma(\wedge^2 V \otimes \mathcal{O}_{\mathbf{P}^6}(1))$ . This seems likely, but we don't know how to prove it.

### 2.3. The case $\mathbf{P}^7$ .

As we saw, for any  $n \leq 6$ , a general s.a. set of points in  $\mathbf{P}^n$  is a hyperplane section of a canonical curve of genus  $n + 2$  in  $\mathbf{P}^{n+1}$ . Clearly, this pattern cannot continue for  $n$  arbitrary large, simply because  $\dim \mathcal{M}_g = 3g - 3$  and  $\dim \mathcal{A}_{g-2} = \binom{g-1}{2}$ . In fact,  $n = 7$  is the first case when a general s.a. set of points in  $\mathbf{P}^7$  is *not* a section of a canonical curve.

To see this, recall Mukai's result that a general canonical curve of genus 9 in  $\mathbf{P}^8$  is a linear section of the isotropic Grassmanian  $G_\omega(3, 6) \subset \mathbf{P}^{13}$ . We claim, that a general s.a. set of points  $\Gamma \subset \mathbf{P}^7$  is *not* a section of  $G_\omega(3, 6)$ . For, let  $\Xi \subset G(\mathbf{P}^7, \mathbf{P}^{13})$  be the open subset of linear sections that meet  $G_\omega(3, 6)$  transversally at 16 distinct points. As before, consider the map

$$\sigma : \Xi / Sp(6) \longrightarrow \mathcal{A}_7.$$

By counting moduli, we get

$$\dim \Xi / Sp(6) = 8(13 - 7) - \binom{6+1}{2} = 27,$$

while

$$\dim \mathcal{A}_7 = \binom{7+1}{2} = 28.$$

Therefore,  $\sigma$  cannot be dominant!

Naturally, we make

**Conjecture 2.12.**  *$\sigma$  is generically injective.*

If the conjecture is true, it would imply that there is a divisor  $\mathcal{Y} \subset \mathcal{A}_7$  parametrizing those s.a. sets of points in  $\mathbf{P}^7$  that arise as a hyperplane sections of canonical curves in  $\mathbf{P}^8$ .

**Remark 2.13.** We note a parallel with the question of whether a given canonical curve  $C \subset \mathbf{P}^{g-1}$  arises as a hyperplane section of a polarized K3 surface ([5],[17],[20]). It is well-known, that if  $g \leq 11$ ,  $g \neq 10$ , then any canonical curve has this property. On the other hand, if  $g = 10$ , then the curves with this property are parametrized by a divisor  $\mathcal{K} \subset \mathcal{M}_{10}$  ([10]).



**APPENDIX: Some computations on Macaulay 2.**

Below are listed the Betti numbers of  $LG_+(5, 10) \subset \mathbf{P}^{15}$  and  $G(2, 6) \subset \mathbf{P}^{14}$  and the schemes of their first infinitesimal neighborhoods. In particular, we see that all schemes in consideration are ACM.

degree						
0	1	—	—	—	—	—
1	—	10	16	—	—	—
2	—	—	—	16	10	—
3	—	—	—	—	—	1

 $LG_+(5, 10)$ 

degree							
0	1	—	—	—	—	—	—
1	—	15	35	21	—	—	—
2	—	—	—	21	35	15	—
3	—	—	—	—	—	—	1

 $G(2, 6)$ 

degree						
0	1	—	—	—	—	—
1	—	—	—	—	—	—
2	—	—	—	—	—	—
3	—	54	144	120	—	—
4	—	—	—	—	45	16

 $LG_+(5, 10)^{(2)}$ 

degree							
0	1	—	—	—	—	—	—
1	—	—	—	—	—	—	—
2	—	1	—	—	—	—	—
3	—	105	399	595	405	105	—
4	—	—	—	—	21	35	15

 $G(2, 6)^{(2)}$

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